

On the heat flux and entropy produced by thermal fluctuations

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We report an experimental and theoretical analysis of the energy exchanged between two conductors kept at different temperature and coupled by the electric thermal noise. Experimentally we determine, as functions of the temperature difference, the heat flux, the out-of-equilibrium variance and a conservation law for the fluctuating entropy, which we justify theoretically. The system is ruled by the same equations of two Brownian particles kept at different temperatures and coupled by an elastic force. Our results set strong constraints on the energy exchanged between coupled nano-systems kept at different temperature.

The fluctuations of thermodynamics variables play an important role for understanding the out-of-equilibrium dynamics of small systems [1, 2], such as Brownian particles [3–6], molecular motors [7] and small devices [8]. The statistical properties of work, heat and entropy, have been analyzed, within the context of fluctuation theorem [9] and stochastic thermodynamics [1, 2], in several experiments on systems in contact with a single heat bath and driven out-of-equilibrium by external forces or fields [3–8]. Instead the important case in which the system is driven out-of-equilibrium by a temperature difference and the energy exchanges are produced only by the thermal noise has been analyzed only theoretically on model systems [10–14] but never in an experiment because of the intrinsic difficulties of dealing with large temperature differences in small systems.

We report here an experimental and theoretical analysis of the statistical properties of the energy exchanged between two conductors kept at different temperature and coupled by the electric thermal noise. This system is inspired by the proof developed by Nyquist [15] in order to give a theoretical explanation of the measurements of Johnson [16] on the thermal noise voltage in conductors. In his proof, assuming thermal equilibrium between the two conductors, he deduces the Nyquist noise spectral density, which at those time, well before Fluctuation Dissipation Theorem (FDT), was the second example, after the Einstein relation for Brownian motion, relating the dissipation of a system to the amplitude of the thermal noise. In this letter we analyze the consequences of removing the Nyquist's equilibrium conditions and we study the statistical properties of the energy exchanged between the two conductors kept at different temperature. This system is probably among the simplest examples where recent ideas of stochastic thermodynamics can be tested but in spite of its simplicity the explanation of the observations is far from trivial. Experimentally we determine, as function of the temperature difference, the heat flux, the out-of-equilibrium variance and a conservation law for the fluctuating entropy, which we justify theoretically. This system is very general because it is ruled by the same equations of two Brownian particles kept at different temperatures and coupled by an elastic force [11, 12]. Thus it gives more insight into the properties of the heat flux, produced by mechanical coupling, in the famous Feynman ratchet [17, 18] widely studied theoretically [12] but never in an experiment. Our results set strong constraints on the energy exchanged between coupled nano-systems kept at different temperature.

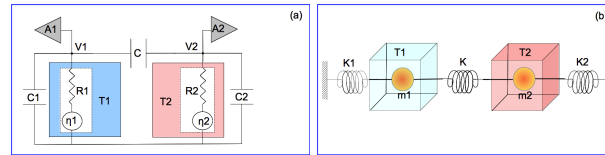


FIG. 1: a) Diagram of the circuit. The resistances R_1 and R_2 are kept at temperature T_1 and $T_2 = 296K$ respectively. They are coupled via the capacitance C . The capacitances C_1 and C_2 schematize the capacitance of the cables and of the amplifiers inputs. The voltages V_1 and V_2 are amplified by the two low noise amplifiers A_1 and A_2 [27]. b) The circuit in a) is equivalent to two Brownian particles (m_1 and m_2) moving inside two different heat baths at T_1 and T_2 . The two particles are trapped by two elastic potentials of stiffness K_1 and K_2 and coupled by a spring of stiffness K (see text and eqs. 3,4). The analogy with the Feynman ratchet can be made by assuming as done in ref. [12] that the particle m_1 has an asymmetric shape and in average moves faster in one direction than in the opposite one.

The system we consider is sketched in fig. 1a). It is constituted by two resistances R_1 and R_2 , which are kept at

different temperature T_1 and T_2 respectively. These temperatures are controlled by thermal baths and T_2 is kept fixed at $296K$ whereas T_1 can be set at a value between $296K$ and $88K$ using a circulation of liquid nitrogen vapors. In the figure, the two resistances have been drawn with their associated thermal noise generators η_1 and η_2 , whose power spectral density are given by the Nyquist formula $|\tilde{\eta}_i|^2 = 4k_B R_i T_i$ (see eqs.3,4 and ref.[20]). The coupling capacitance C controls the electrical power exchanged between the resistances and as a consequence the energy exchanged between the two baths. The quantities C_1 and C_2 are the capacitances of the circuits and the cables. Two extremely low noise amplifiers A_1 and A_2 [27] measure the voltage V_1 and V_2 across the resistances R_1 and R_2 respectively. When $T_1 = T_2$ the system is in equilibrium and exhibits no net energy flux between the two reservoirs. This is indeed the condition imposed by Nyquist to prove his formula, and we use it to calibrate the system and check the values of the capacitances applying Fluctuation-Dissipation-Theorem (FDT) to the circuit (see ref.[20] for the calibration). In the following we will take $C = 100pF$, $C_1 = 680pF$, $C_2 = 420pF$ and $R_1 = R_2 = 10M\Omega$, if not differently stated.

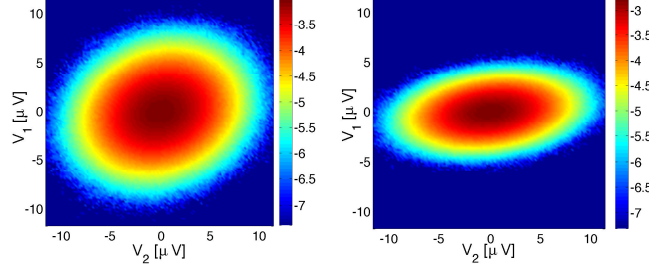


FIG. 2: The joint probability $\log_{10} P(V_1, V_2)$ measured at $T_1 = 296K$ equilibrium (a) and out of equilibrium $T_1 = 88K$ (b). The color scale is indicated on the colorbar on the right side.

The important quantity to consider here is the joint probability $P(V_1, V_2)$, which is plotted in fig. 2a) at $T_1 = T_2$ and at fig. 2b) at $T_1 = 88K$. The fact that the axis of the ellipses defining the contours lines of $P(V_1, V_2)$ are inclined with respect to the x and y axis indicates that there is a certain correlation between V_1 and V_2 . This correlation, produced by the electric coupling, plays a major role in determining the mean heat flux between the two reservoirs, as we will discuss in the following. The interesting new features occur of course when $T_1 \neq T_2$. The questions that we address for such a system are: What are the heat flux and the entropy production rate ? How the variance of V_1 and V_2 are modified because of the heat flux ? What is the role of correlation between V_1 and V_2 ? We will see that these questions are quite relevant and have no obvious answers because of the statistical nature of the energy transfer.

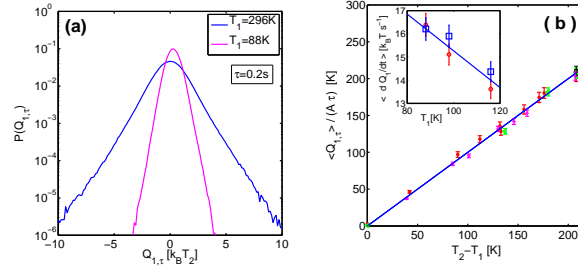


FIG. 3: a) The probability $P(Q_{1,\tau})$ measured at $T_1 = 296K$ (blue line) equilibrium and $T_1 = 88K$ (magenta line) out of equilibrium. Notice that the peak of the $P(Q_{1,\tau})$ is centered in zero at equilibrium and shifted towards positive value out of equilibrium. The amount of the shift is very small and it is about $k_B T_2$ b) The measured mean value of $\langle Q_{1,\tau} \rangle$ is a linear function of $(T_2 - T_1)$. The red points correspond to measures performed with the values of the capacitance C_1, C_2, C given in the text and $\tau = 0.2s$. The other symbols and colors pertain to different values of these capacitance and other τ . The values of $\langle Q_{1,\tau} \rangle$ have been rescaled by the parameter dependent theoretical prefactor A , which allows the comparison of different experimental configurations. The continuous blu line with slope 1 is the theoretical prediction (see eq. 7) In the inset the values of $\langle \dot{Q}_1 \rangle$ (at $C = 1000pF$) directly measured using $P(Q_1)$ (blue square) are compared with those (red circle) obtained from the equality $\langle \dot{Q}_1 \rangle = (\sigma_1^2 - \sigma_{1,eq}^2)/R_1$, as discussed in the text.

We consider the electric power dissipated in the resistance R_m with $m = 1, 2$ which reads $\dot{Q}_m = V_m i_m$ where i_m is the current flowing in the resistance m . The integral of the power over a time τ is the total energy Q_m , dissipated

by the resistance in this time interval, i.e. $Q_{m,\tau} = \int_t^{t+\tau} i_m V_m dt$. All the V_m and i_m can be measured indeed $i_m = i_C - i_{C_m}$ where $i_C = C \frac{d(V_2 - V_1)}{dt}$ is the current flowing in the capacitance C and $i_{C_m} = C_m \frac{dV_m}{dt}$ the current flowing in C_m . Thus rearranging the terms one finds that $Q_{m,\tau} = W_{m,\tau} - \Delta U_{m,\tau}$ where $W_{1,\tau} = \int_t^{t+\tau} C V_1 \frac{dV_2}{dt} dt$, $W_{2,\tau} = \int_t^{t+\tau} C V_2 \frac{dV_1}{dt} dt$ and $\Delta U_{m,\tau} = \frac{(C_m + C)}{2} (V_m(t + \tau)^2 - V_m(t)^2)$ is the potential energy change of the circuit m in the time τ . Notice that W_m are the terms responsible for the energy exchanges as they couple the fluctuations of the two circuits. The quantities $W_{1,\tau}$ and $W_{2,\tau}$ can be identified as the work performed by the circuit 2 on 1 and vice-versa [19, 21, 24], respectively. Thus, the quantity $Q_{1,\tau}$ ($Q_{2,\tau}$) can be interpreted as the heat flowing from the reservoir 2 to the reservoir 1 (from 1 to 2), in the time interval τ , as an effect of the temperature difference. As the two variables V_m are fluctuating voltages all the other quantities fluctuate too. In fig. 3a) we show the probability density function $P(Q_{1,\tau})$, at various temperatures: we see that $Q_{1,\tau}$ is a strongly fluctuating quantity, whose PDF has long exponential tails.

Notice that although for $T_1 < T_2$ the mean value of $Q_{1,\tau}$ is positive, instantaneous negative fluctuations can occur, i.e., sometimes the heat flux is reversed. The mean values of the dissipated heats are expected to be linear functions of the temperature difference $\Delta T = T_2 - T_1$, i.e. $\langle Q_{1,\tau} \rangle = A \tau \Delta T$, where A is a parameter dependent quantity, that can be obtained explicitly from eqs. 3 and 4, as discussed in ref.[20]. This is confirmed by our results, as shown in fig. 3b). Furthermore, the mean values of the dissipated heat satisfy the equality $\langle Q_2 \rangle = -\langle Q_1 \rangle$, corresponding to an energy conservation principle: the power extracted from the bath 2 is dissipated into the bath 1 because of the electric coupling. This mean flow produces a change of the variances σ_m^2 of V_m with respect to the equilibrium value $\sigma_{m,eq}^2$, i.e. the Nyquist value measured at the same T_m but at $\Delta T = 0$. Specifically we find $\sigma_m^2 = \sigma_{m,eq}^2 + \langle \dot{Q}_m \rangle R_1$ (see ref.[20]) which is an extension to the two temperatures case of the Harada-Sasa relation [28]. This result is shown in the inset of fig. 3b) where the values of $\langle \dot{Q}_m \rangle$ directly estimated from the experimental data (using the steady state $P(Q_m)$) are compared with those obtained from the difference of the variances of V_1 measured in equilibrium and out-of-equilibrium. The values are comparable within error bars and show that the out-of-equilibrium variances are modified just by the heat flux.

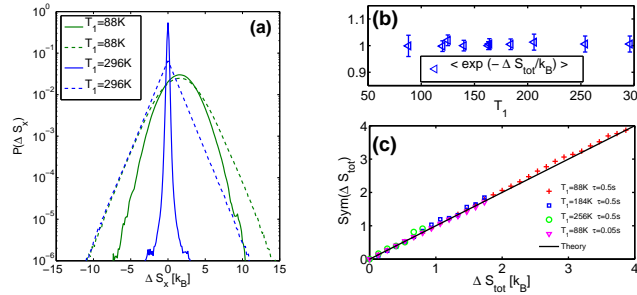


FIG. 4: a) The probability $P(\Delta S_r)$ (dashed lines) and $P(\Delta S_{tot})$ (continuous lines) measured at $T_1 = 296K$ (blue line) equilibrium and $T_1 = 88K$ (green lines) out of equilibrium. Notice that both distributions are centered in zero at equilibrium and shifted towards positive value in the out-of-equilibrium. b) $\langle \exp(-\Delta S_{tot}) \rangle$ as a function of T_1 at two different $\tau = 0.5s$ and $\tau = 0.1s$. c) Symmetry function $Sym(\Delta S_{tot}) = \log[P(\Delta S_{tot})/P(-\Delta S_{tot})]$ as a function of ΔS_{tot} . The black straight line of slope 1 corresponds to the theoretical prediction.

It is now important to analyze the entropy produced by the total system, circuit plus heat reservoirs. We consider first the entropy $\Delta S_{r,\tau}$ due to the heat exchanged with the reservoirs, which reads $\Delta S_{r,\tau} = Q_{1,\tau}/T_1 + Q_{2,\tau}/T_2$. This entropy is a fluctuating quantity as both Q_1 and Q_2 fluctuate, and its average in a time τ is $\langle \Delta S_{r,\tau} \rangle = \langle Q_{r,\tau} \rangle (1/T_1 - 1/T_2) = A \tau (T_2 - T_1)^2 / (T_2 T_1)$. However the reservoir entropy $\Delta S_{r,\tau}$ is not the only component of the total entropy production: one has to take into account the entropy variation of the system, due to its dynamical evolution. Indeed, the state variables V_m also fluctuate as an effect of the thermal noise, and thus, if one measures their values at regular time interval, one obtains a “trajectory” in the phase space $(V_1(t), V_2(t))$. Thus, following Seifert [22], who developed this concept for a single heat bath, one can introduce a trajectory entropy for the evolving system $S_s(t) = -k_B \log P(V_1(t), V_2(t))$, which extends to non-equilibrium systems the standard Gibbs entropy concept. Therefore, when evaluating the total entropy production, one has to take into account the contribution

$$\Delta S_{s,\tau} = -k_B \log \left[\frac{P(V_1(t + \tau), V_2(t + \tau))}{P(V_1(t), V_2(t))} \right]. \quad (1)$$

It is worth noting that the system we consider is in a non-equilibrium steady state, with a constant external driving

ΔT . Therefore the probability distribution $P(V_1, V_2)$ (as shown in fig. 2b)) does not depend explicitly on the time, and $\Delta S_{s,\tau}$ is non vanishing whenever the final point of the trajectory is different from the initial one: $(V_1(t+\tau), V_2(t+\tau)) \neq (V_1(t), V_2(t))$. Thus the total entropy reads $\Delta S_{tot,\tau} = \Delta S_{r,\tau} + \Delta S_{s,\tau}$, where we omit the explicit dependence on t , as the system is in a steady-state as discussed above. This entropy has several interesting features. The first one is that $\langle \Delta S_{s,\tau} \rangle = 0$, as a consequence $\langle \Delta S_{tot} \rangle = \langle \Delta S_r \rangle$ which grows with increasing ΔT . The second and most interesting result is that independently of ΔT and of τ , the following equality always holds:

$$\langle \exp(-\Delta S_{tot}/k_B) \rangle = 1, \quad (2)$$

for which we find both experimental evidence, as discussed in the following, and provide a theoretical proof, as discussed in detail in ref. [20]. Indeed, eq. (2) can be proved by relating the reservoir entropy variation $\Delta S_{r,\tau}$ to the system single trajectory probability. Equation (2) represents an extension to two temperature sources of the result obtained for a system in a single heat bath driven out-of-equilibrium by a time dependent mechanical force [6, 22] and is the first experimental result obtained on a system driven by a temperature difference. Eq. (2) implies that $\langle \Delta S_{tot} \rangle \geq 0$, as prescribed by the second principle. The other consequence of eq. (2) is that ΔS_{tot} takes into account only the out-of-equilibrium fluctuations. Indeed its PDF is a delta function centered in zero when $T_1 = T_2$.

The measured probabilities $P(\Delta S_r)$ and $P(\Delta S_{tot})$ are shown in fig. 4a). We see that $P(\Delta S_r)$ and $P(\Delta S_{tot})$ are quite different and that the latter is close to a Gaussian and reduces to a Dirac δ function in equilibrium, i.e. $T_1 = T_2 = 296K$ (notice that the small broadening observed in equilibrium is just due to unavoidable experimental noise and discretization of the experimental PDF). Thus we find here a quantity that is rigorously zero in equilibrium, both in average and fluctuations, and its mean value and variance provide a measure of the entropy production. The experimental measurements satisfy eq.2 as it is shown in fig. 4b). It is worth to note that eq.2 implies that $P(\Delta S_{tot})$ should satisfy a fluctuation theorem of the form $\log[P(\Delta S_{tot})/P(-\Delta S_{tot})] = \Delta S_{tot}/k_B$, $\forall \tau, \Delta T$, as discussed extensively in reference [1, 23]. We clearly see in fig.4c) that this relation holds for different values of the temperature gradient. Thus this experiment clearly establishes a relationship between the mean and the variance of the entropy production rate in a system driven out-of-equilibrium by the temperature difference between two thermal baths coupled by electric noise. Because of the formal analogy with Brownian motion the results also apply to mechanical coupling as discussed in the following.

For the experimental analysis we have not used any theoretical input and we will now give a theoretical interpretation of the experimental observations. This will allow us to show the analogy of our system with two interacting Brownian particles coupled to two different temperatures, see fig. 1-b). Let q_m ($m = 1, 2$) be the charges that have flowed through the resistances R_m , so the instantaneous current flowing through them is $i_m = \dot{q}_m$. A circuit analysis shows that the equations for the charges are:

$$R_1 \dot{q}_1 = -q_1 \frac{C_2}{X} + (q_2 - q_1) \frac{C}{X} + \eta_1 \quad (3)$$

$$R_2 \dot{q}_2 = -q_2 \frac{C_1}{X} + (q_1 - q_2) \frac{C}{X} + \eta_2 \quad (4)$$

where $X = C_2 C_1 + C(C_1 + C_2)$ and η_m is the usual white noise: $\langle \eta_i \eta_j' \rangle = 2\delta_{ij} k_B T_i R_j \delta(t - t')$. The relationships between the measured voltages and the charges are :

$$q_1 = (V_1 - V_2) C + V_1 C_1 \quad (5)$$

$$q_2 = (V_1 - V_2) C - V_2 C_2 \quad (6)$$

Eqs. 3 and 4 are the same of those for the two coupled Brownian particles sketched in fig.1b) by considering q_m the displacement of the particle m , i_m its velocity, $K_m = 1/C_m$ the stiffness of the spring m and $K = 1/C$ the coupling spring. With this analogy we see that our definition of Q_m corresponds exactly to the work performed by the viscous forces and by the bath on the particle m , and it is consistent with the stochastic thermodynamics definition [1, 19, 24–26].

Thus our theoretical analysis and the experimental results apply to both interacting mechanical and electrical systems coupled to baths at different temperatures. Starting from eqs. (3)-(4), we can prove (see ref. [20]) that eq.2 is an exact result and that the average dissipated heat rate is

$$\langle \dot{Q}_1 \rangle = A(T_2 - T_1) = \frac{C^2 \Delta T}{XY}, \quad (7)$$

with $Y = [(C_1 + C)R_1 + (C_2 + C)R_2]$ and $A = C^2/(XY)$ is the parameter used to rescale the data in fig. 3b).

To conclude we have studied experimentally the statistical properties of the energy exchanged between two heat baths at different temperature which are coupled by electric thermal noise. We have measured the heat flux, the entropy production rate and we have shown the existence of a conservation law for entropy which imposes the existence of a fluctuation theorem which is not asymptotic. Our results, which are theoretically proved, are very general since the electric system considered here is ruled by the same equations as for two Brownian particles, kept at different temperatures and mechanically coupled. Therefore these results set precise constraints on the energy exchanged between coupled nano and micro-systems kept at different temperature. We finally mention that for the quantity W_i an asymptotic FT can be proved both experimentally and theoretically, and this will be the subject of a paper in preparation.

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- [1] U. Seifert *Rep. Progr. Phys.* **75**, 126001, (2012).
 - [2] Sekimoto, K. *Stochastic Energetics*. (Springer, 2010).
 - [3] Blickle V., Speck T., Helden L., Seifert U., and Bechinger C. *Phys. Rev. Lett.* **96**, 070603 (2006).
 - [4] Jop, P., Petrosyan A. and Ciliberto S. *EPL* **81**, 50005 (2008).
 - [5] Gomez-Solano, J. R., Petrosyan, A., Ciliberto, S. Chetrite, R. and Gawedzki K. *Phys. Rev. Lett* 103, 040601 (2009).
 - [6] G. M. Wang, E. M. Sevick, E. Mittag, D. J. Searles, and D. J. Evans, *Phys. Rev. Lett.*, **89**: 050601 (2002).
 - [7] K. Hayashi, H. Ueno, R. Iino, H. Noji, *Phys. Rev. Lett.* 104, 218103 (2010)
 - [8] S Ciliberto, S Joubaud and A Petrosyan *J. Stat. Mech.*, P12003 (2010).
 - [9] D. J. Evans *et al.*, *Phys. Rev. Lett.* **71**, 2401 (1993); G. Gallavotti, E. G. D. Cohen, *J. Stat. Phys.* **80**, 931 (1995).
 - [10] T. Bodinau, B. Derida, *Phys. Rev. Lett* 92, 180601 (2004).
 - [11] A. Crisanti, A. Puglisi, and D. Villamaina, *Phys. Rev. E* 85, 061127 (2012)
 - [12] C. Van den Broeck, R. Kawai and P. Meurs, *Phys. Rev. Lett* 93, 090601 (2004).
 - [13] P Visco, *J. Stat. Mech.*, page P06006, (2006)
 - [14] Evans D. , Searles D. J. Williams S. R., *J. Chem. Phys.* 132 0245012010.
 - [15] H. Nyquist, *Phys. Rev.* 32, 110 (1928)
 - [16] J. Johnson, *Phys. Rev.* 32, 97 (1928)
 - [17] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics I* (Addison-Wesley, Reading, MA, 1963), Chap. 46.
 - [18] M. v. Smoluchowski, *Phys. Z.* 13, 1069 (1912).
 - [19] Sekimoto K, *Prog. Theor. Phys. Suppl.* 130, 17, (1998)
 - [20] [Supplementary Material](#).
 - [21] N. Garnier, S. Ciliberto, *Phys. Rev. E*, 71, 060101(R) (2005)
 - [22] U. Seifert, *Phys. Rev. Lett.*, **95**: 040602 (2005).
 - [23] M. Esposito, C. Van den Broeck, *Phys. Rev. Lett.*, **104**, 090601 (2010).
 - [24] R. van Zon, S. Ciliberto, E. G. D. Cohen, *Phys. Rev. Lett.* **92**: 130601 (2004).
 - [25] A. Imparato, L. Peliti, G. Pesce, G. Rusciano, A. Sasso, *Phys. Rev. E*, **76**: 050101R (2007).
 - [26] H. C. Fogedby, A. Imparato, *J. Stat. Mech.* P04005 (2012).
 - [27] G. Cannatà, G. Scandurra, C. Ciofin, *Rev. Scie. Instrum.* 80, 114702 (2009).
 - [28] Harada T. Sasa S.-I., *Phys. Rev. Lett.*, 95, 130602 (2005).

On the heat flux and entropy produced by thermal fluctuations: Supplementary information

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I. EXPERIMENTAL DETAILS

A. Experimental set up

The electric systems and amplifiers are inside a Faraday cage and mounted on a floating optical table to reduce mechanical and acoustical noise. The resistance R_1 , which is cooled by liquid Nitrogen vapors, changes of less than 0.1% in the whole temperature range. Its temperature is measured by a PT1000 which is inside the same shield of R_1 . The signal V_1 and V_2 are amplified by two custom designed JFET amplifiers [1] with an input current of $1pA$ and a noise of $0.7nV\sqrt{Hz}$ at frequencies larger than $1Hz$ and increases at $8nV\sqrt{Hz}$ at $0.1Hz$. The resistances $R1$ and $R2$ have been used as input resistances of the amplifiers. The two signals V_1 and V_2 are amplified 10^4 times and the amplifiers outputs are filtered (at $4KHz$ to avoid aliasing) and acquired at $8KHz$ by a 24 bits by an ADC. We used different sets of C_1 , C_2 and C . The values of $C1$ and $C2$ are essentially set by the input capacitance of the amplifiers and by the cable length $680pF < C1 < 780pF$ and $400pF < C2 < 500pF$. Instead C has been changed from $100pF$ to $1000pF$. The system has always been calibrated in equilibrium at $T_1 = T_2 = 296K$ using the FDT and estimating the spectrum using the values of the capacitances, see next sections.

B. Noise spectrum of the amplifiers

The noise spectrum of the amplifiers A_1 and A_2 (Fig.1 of the main text), measured with a short circuit at the inputs, is plotted in fig.S.1a) and compared with the spectrum Sp_1 of V_1 at $T_1 = 88$. We see that the useful signal is several order of magnitude larger than the amplifiers noise.

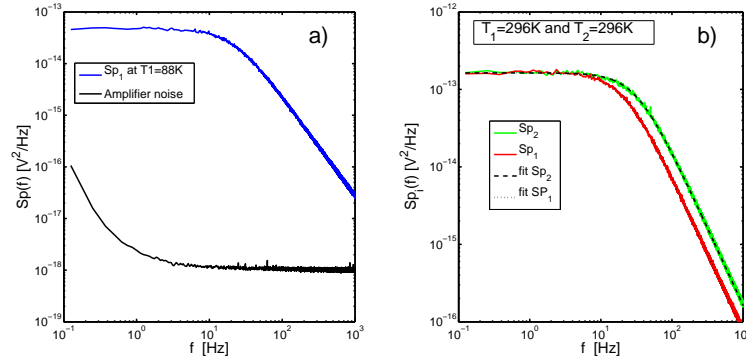


FIG. S.1: a) The power spectra Sp_1 of V_1 measured at $T_1 = 88K$ (blue line) ($C = 100pF$, $C_1 = 680pF$, $C_2 = 430pF$) is compared to the spectrum of the amplifier noise. b) The equilibrium spectra Sp_1 (red line) and Sp_2 (green line) measured at $T_1 = T_2 = 296K$ are compared with prediction of eqs.S.1 and S.2 in order to check the values of the capacitances (C_1 , C_2).

C. Check of the calibration

The equilibrium spectra of V_1 and V_2 at $T_1 = T_2$ used for calibration of the capacitances are:

$$S_1(\omega) = \frac{4k_B T_1 R_1 [1 + \omega^2 (C^2 R_1 R_2 + R_2^2 (C_2 + C)^2)]}{(1 - \omega^2 X R_1 R_2)^2 + \omega^2 Y} \quad (\text{S.1})$$

$$S_2(\omega) = \frac{4k_B T_2 R_1 [1 + \omega^2 (C^2 R_1 R_2 + R_2^2 (C_1 + C)^2)]}{(1 - \omega^2 X R_1 R_2)^2 + \omega^2 Y} \quad (\text{S.2})$$

This spectra can be easily obtained by applying FDT to the circuit of fig.1 in the main text.

The two computed spectra are compared to the measured ones in fig. S.1a). This comparison allows us to check the values of the capacitances C_1 and C_2 which depend on the cable length. We see that the agreement between the prediction and the measured power spectra is excellent and the global error on calibration is of the order of 1%. This corresponds exactly to the case discussed by Nyquist in which the two resistances at the same temperature are exchanging energy via an electric circuit (C in our case).

D. The power spectra of V_1 and V_2 out-of-equilibrium

When $T_1 \neq T_2$ the power spectra of V_1 and V_2 are:

$$S_1(\omega) = \frac{4k_B T_1 R_1 [1 + \omega^2 (C^2 R_1 R_2 + R_2^2 (C_2 + C)^2)]}{(1 - \omega^2 X R_1 R_2)^2 + \omega^2 Y} + \frac{4k_B (T_2 - T_1) \omega^2 C^2 R_1^2 R_2}{(1 - \omega^2 X R_1 R_2)^2 + \omega^2 Y} \quad (\text{S.3})$$

$$S_2(\omega) = \frac{4k_B T_2 R_1 [1 + \omega^2 (C^2 R_1 R_2 + R_2^2 (C_1 + C)^2)]}{(1 - \omega^2 X R_1 R_2)^2 + \omega^2 Y} + \frac{4k_B (T_1 - T_2) \omega^2 C^2 R_2^2 R_1}{(1 - \omega^2 X R_1 R_2)^2 + \omega^2 Y} \quad (\text{S.4})$$

These equations have been obtained by Fourier transforming eqs. S.7,S.8, solving for $\tilde{V}_1(\omega)$ and $\tilde{V}_2(\omega)$ and computing the modula. The integral of eqs. S.3 and S.4 gives the variances eq. S.24 directly computed from the distributions.

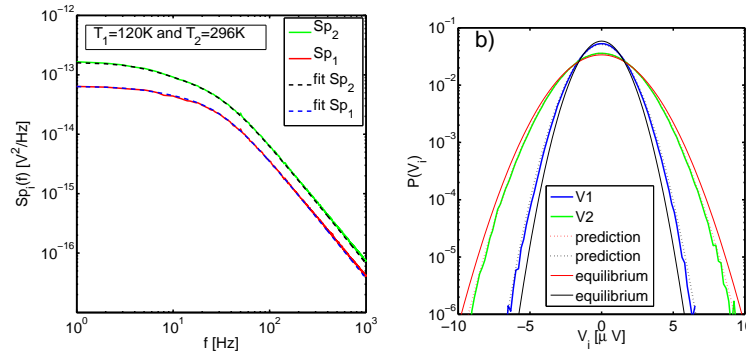


FIG. S.2: a) The power spectra Sp_1 of V_1 and Sp_2 of V_2 measured at $T_1 = 120K$ and $T_2 = 296K$ ($C = 100pF$, $C_1 = 680pF$, $C_2 = 430pF$) are compared with the prediction of eq.S.3 and S.4 (dashed lines) b) The corresponding Probability Density Function $P(V_1)$ of V_1 (green line) and $P(V_2)$ of V_2 (blue line) measured at $T_1 = 120K$ and $T_2 = 296K$. Dotted lines are the out-of-equilibrium PDF, whose variance is estimated from the measure of the heat flux (see fig.3 in the main text) and eq.S.24 in the following section. The continuous red line is the equilibrium $P(V_2)$ at $T_1 = 296K$ and the black continuous line corresponds to the equilibrium $P(V_1)$ at $T_2 = 120K$.

E. Measure of the equilibrium variance of V_1 as a function of T_1

This measure is necessary to estimate $\langle Q_1 \rangle$ starting from the measurement of the variances as explained in fig.4 of the main text. We first measure $\sigma_{m,eq}^2(T_1)$ at $T_1 = T_2 = 296K$. In this way one gets the proportionality constant $\alpha_1 = \sigma_{m,eq}^2(T_1 = T_2)/T_2$ as in equilibrium the variance must be proportional to T_1 . Thus when $T_1 < T_2$ one gets $\sigma_{m,eq}^2(T_1) = \alpha_1 T_1$. As

explained in the main text $\langle \dot{Q}_1 \rangle = (\sigma_1^2(T1) - \sigma_{m,eq}^2(T1))/R_1$. In fig. S.2b) we compare the measured PDF of V_1 and V_2 with the equilibrium and the out-of-equilibrium distributions as computed by using the theoretical predictions eq.S.24 for the variance.

II. DYNAMICAL EQUATIONS FOR V_m AND Q_m .

We want to describe the dynamical evolution of both the electric and thermodynamic variables for the circuit in fig. 1 in the main text with a set of coupled Langevin equations. For this purpose we write the Langevin equations governing the dynamical evolutions for the voltages across the circuit:

$$(C_1 + C)\dot{V}_1 = C\dot{V}_2 + \frac{1}{R_1}(\eta_1 - V_1) \quad (S.5)$$

$$(C_2 + C)\dot{V}_2 = C\dot{V}_1 + \frac{1}{R_2}(\eta_2 - V_2) \quad (S.6)$$

where we have substituted eqs. (5)-(6) into eqs. (3)-(4) in the main text. We rearrange these equations in a standard form, and obtain

$$\dot{V}_1 = f_1(V_1, V_2) + \sigma_{11}\eta_1 + \sigma_{12}\eta_2 = f_1(V_1, V_2) + \xi_1 \quad (S.7)$$

$$\dot{V}_2 = f_2(V_1, V_2) + \sigma_{21}\eta_1 + \sigma_{22}\eta_2 = f_2(V_1, V_2) + \xi_2 \quad (S.8)$$

where the “forces” acting on the circuits read

$$f_1(V_1, V_2) = \alpha_1 V_1 + \alpha_2 V_2 = -\frac{C_2 R_2 V_1 + C(R_2 V_1 + R_1 V_2)}{[C_2 C + C_1(C_2 + C)]R_1 R_2}, \quad (S.9)$$

$$f_2(V_1, V_2) = \gamma_1 V_1 + \gamma_2 V_2 = -\frac{C_1 R_1 V_2 + C(R_2 V_1 + R_1 V_2)}{[C_2 C + C_1(C_2 + C)]R_1 R_2}, \quad (S.10)$$

the coefficients σ_{ij} read

$$\begin{aligned} \sigma_{11} &= \frac{C_2 + C}{X R_1} \\ R_2 \sigma_{12} &= R_1 \sigma_{21} = \frac{C}{X} \\ \sigma_{22} &= \frac{C_1 + C}{X R_2}, \end{aligned}$$

and the noises ξ_i introduced in eqs. (S.7)-(S.8) are now correlated $\langle \xi_i \xi_j' \rangle = 2\theta_{ij}\delta(t - t')$, where

$$\theta_{11} = \frac{T_1(C_2 + C)^2}{R_1(C_2 C + C_1(C_2 + C))^2} + \frac{T_2 C^2}{R_2(C_2 C + C_1(C_2 + C))^2}, \quad (S.11)$$

$$\theta_{12} = \frac{T_1(C(C_2 + C))}{R_1(C_2 C + C_1(C_2 + C))^2} + \frac{T_2(C(C_1 + C))}{R_2(C_2 C + C_1(C_2 + C))^2}, \quad (S.12)$$

$$\theta_{22} = \frac{T_1 C^2}{R_1(C_2 C + C_1(C_2 + C))^2} + \frac{T_2(C_1 + C)^2}{R_2(C_2 C + C_1(C_2 + C))^2}, \quad (S.13)$$

and $\theta_{12} = \theta_{21}$. We now notice that the rate of the dissipated heat in circuit m reads

$$\dot{Q}_m = V_m i_m = \frac{V_m}{R_m}(V_m - \eta_m) = (C_m + C)\dot{V}_1 - C\dot{V}_{m'}, \quad (S.14)$$

where $m' = 2$ if $m = 1$, and $m' = 1$ if $m = 2$. The rightmost equality in eq. (S.14) follows immediately from eqs. (S.5)-(S.6). So one has a formalism where both the voltages and the dissipated heats are described as stochastic processes, driven by the thermal noises η_m .

III. PROBABILITY DISTRIBUTION FUNCTION FOR THE VOLTAGES

We now study the joint probability distribution function (PDF) $P(V_1, V_2, t)$, that the system at time t has a voltage drop V_1 across the resistor R_1 and a voltage drop V_2 across the resistor R_2 . As the time evolution of V_1 and V_2 is described by the Langevin equations (S.7)-(S.8), it can be proved that the time evolution of $P(V_1, V_2, t)$ is governed by the Fokker-Planck equation [2]

$$\begin{aligned} \partial_t P(V_1, V_2, Q_1, t) = L_0 P(V_1, V_2, t) = & -\frac{\partial}{\partial V_1} (f_1 P) - \frac{\partial}{\partial V_2} (f_2 P) + \theta_{11} \frac{\partial^2}{\partial V_1^2} P + \theta_{22} \frac{\partial^2}{\partial V_2^2} P \\ & + 2\theta_{12} \frac{\partial^2}{\partial V_1 \partial V_2} P \end{aligned} \quad (\text{S.15})$$

We are interested in the long time steady state solution of eq. (S.15), which is time independent $P(V_1, V_2, t \rightarrow \infty) = P_{ss}(V_1, V_2)$. As the deterministic forces in eqs. (S.7)-(S.8) are linear in the variables V_1 and V_2 , such a steady state solution reads

$$P_{ss}(V_1, V_2) = \frac{2\pi e^{-(aV_1^2 + bV_1V_2 + cV_2^2)}}{\sqrt{-b^2 + 4ca}} \quad (\text{S.16})$$

where the coefficients

$$\begin{aligned} a &= \frac{X \{C_1 T_2 Y + C[CR_2 T_1 + T_2(C_1 R_1 + CR_1 + C_2 R_2)]\}}{2[Y^2 T_1 T_2 + C^2 R_1 R_2 (T_1 - T_2)^2]}, \\ b &= -\frac{XC[(C_2 + C)R_2 T_1 + (C_1 + C)R_1 T_2]}{[Y^2 T_1 T_2 + C^2 R_1 R_2 (T_1 - T_2)^2]}, \\ c &= \frac{X \{C_2 T_1 Y + C[CR_1 T_2 + T_1(C_1 R_1 + CR_2 + C_2 R_2)]\}}{2[Y^2 T_1 T_2 + C^2 R_1 R_2 (T_1 - T_2)^2]}, \end{aligned}$$

can be obtained by replacing eq. (S.16) into eq. (S.15), and by imposing the steady state condition $\partial_t P = 0$. We are furthermore interested in the unconstrained steady state probabilities $P_{1,ss}(V_1)$, and $P_{2,ss}(V_2)$, which are obtained as follows

$$P_{1,ss}(V_1) = \int dV_2 P_{ss}(V_1, V_2) = \frac{e^{-\frac{V_1^2}{2\sigma_1^2}}}{\sqrt{2\pi\sigma_1^2}} \quad (\text{S.17})$$

$$P_{2,ss}(V_2) = \int dV_1 P_{ss}(V_1, V_2) = \frac{e^{-\frac{V_2^2}{2\sigma_2^2}}}{\sqrt{2\pi\sigma_2^2}} \quad (\text{S.18})$$

where the variances read

$$\sigma_1^2 = \frac{T_1(C + C_2)Y + (T_2 - T_1)C^2 R_1}{XY} \quad (\text{S.19})$$

$$\sigma_2^2 = \frac{T_2(C + C_1)Y - (T_2 - T_1)C^2 R_2}{XY} \quad (\text{S.20})$$

IV. PROBABILITY DISTRIBUTION FOR THE DISSIPATED HEAT AND AVERAGE RATE

We start by noticing that the heat injected from the bath 1 is then dissipated in the bath 2 (and vice-versa), and so we expect the probability distribution of Q_1 and Q_2 to be symmetric. Thus in the following, we will only study the probability distribution of Q_1 . We now proceed by introducing the joint probability distribution function of the variables V_1 , V_2 , and Q_1 , $\Phi(V_1, V_2, Q_1, t)$. As each of these three variables evolves according to a Langevin equation, the time evolution of their PDF is described by the Fokker-Planck equation [3, 4]

$$\begin{aligned} \partial_t \Phi(V_1, V_2, Q_1, t) = & -\frac{\partial}{\partial V_1} (f_1 \Phi) - \frac{\partial}{\partial V_2} (f_2 \Phi) + \theta_{11} \frac{\partial^2}{\partial V_1^2} \Phi + \theta_{22} \frac{\partial^2}{\partial V_2^2} \Phi + 2\theta_{12} \frac{\partial^2}{\partial V_1 \partial V_2} \Phi \\ & - \frac{\partial}{\partial Q_1} \left\{ r_{11} \left[\frac{\partial}{\partial V_1} (V_1 \Phi) + \left(V_1 \frac{\partial}{\partial V_1} \Phi \right) \right] + 2r_{12} \frac{\partial}{\partial V_2} (V_1 \Phi) + \frac{V_1^2}{R_1} \Phi \right\} \\ & + V_1^2 r_{22} \frac{\partial^2}{\partial Q_1^2} \Phi \end{aligned} \quad (\text{S.21})$$

with

$$\begin{aligned} r_{11} &= k_1\theta_{11} + k_2\theta_{12}, \\ r_{12} &= k_1\theta_{12} + k_2\theta_{22}, \\ r_{22} &= k_1^2\theta_{11} + k_2^2\theta_{22} + 2k_1k_2\theta_{12}, \end{aligned} \quad (\text{S.22})$$

and $k_1 = (C_1 + C)$, $k_2 = -C$. It is worth noting that the first part of the right hand side of eq. (S.21) is identical to the rhs of eq. (S.15).

We proceed by proving eq. (7) in the main text, expressing the dissipated heat rate as a function of the system parameters. We have

$$\begin{aligned} \partial_t \langle Q_1 \rangle_t &= \partial_t \int dV_1 dV_2 dQ_1 Q_1 \Phi(V_1, V_2, Q_1, t) = \int dV_1 dV_2 dQ_1 Q_1 \partial_t \Phi(V_1, V_2, Q, t) = -r_{11} + \frac{1}{R_1} \langle V_1^2 \rangle \\ &= \frac{C^2 \Delta T}{XY}, \end{aligned} \quad (\text{S.23})$$

where we have replaced the time derivative $\partial_t \Phi(V_1, V_2, Q, t)$ with the rhs of eq. (S.21) and used the equality $\langle V_1^2 \rangle = \sigma_1^2$, with σ_1 as given by eq. (S.19). This equation corresponds to the one given in the main text.

We can now obtain the expressions for the variance of V_1 and V_2 , as introduced in the main text. Using eq.(S.23) we can express eq. (S.19) and eq. (S.20) in the following way:

$$\sigma_m^2 = \sigma_{m,\text{eq}}^2 + \langle \dot{Q}_m \rangle R_m \quad (\text{S.24})$$

where $\sigma_{m,\text{eq}}^2 = \frac{T_m(C+C_m)}{X}$ is the equilibrium value of σ_m^2 at $\langle \dot{Q}_m \rangle = 0$.

V. CONSERVATION LAW

We now turn our attention to eq. (2), and provide a formal proof for it. In order to do this, we derive a relation between the reservoir entropy change $\Delta S_{r,\tau}$ and the system dynamics. For simplicity, in the following we divide the time into small intervals Δt : let us assume that the system (the circuit in our case) is in the state $\mathbf{V} = (V_1, V_2)$ at time t , and let's denote by $\mathbf{V}' = (V_1 + \Delta V_1, V_2 + \Delta V_2)$ its state at time $t + \Delta t$. Let $\mathcal{P}_F(\mathbf{V} \rightarrow \mathbf{V}' | \mathbf{V}, t)$ be the probability that the system undergoes a transition from \mathbf{V} to \mathbf{V}' provided that its state at time t is \mathbf{V} , and let $\mathcal{P}_R(\mathbf{V}' \rightarrow \mathbf{V} | \mathbf{V}', t + \Delta t)$ be the probability of the time-reverse transition. We have

$$\begin{aligned} P_F(\mathbf{V} \rightarrow \mathbf{V}' | \mathbf{V}, t) &= \int d\eta_1 d\eta_2 \delta(\Delta V_1 - \Delta t(f_1(V_1, V_2) + \sigma_{11}\eta_1 + \sigma_{12}\eta_2)) \\ &\quad \times \delta(\Delta V_2 - \Delta t(f_2(V_1, V_2) + \sigma_{21}\eta_1 + \sigma_{22}\eta_2)) p_1(\eta_1) p_2(\eta_2), \end{aligned} \quad (\text{S.25})$$

where $\delta(x)$ is the Dirac delta function. Given that the noises are Gaussian distributed, their probability distributions read

$$p_m(\eta_m) = \exp \left[-\frac{\eta_m^2 \Delta t}{4R_m k_B T} \right] \sqrt{\frac{\Delta t}{4\pi R_m k_B T_m}} \quad (\text{S.26})$$

and expressing the Dirac delta in Fourier space $\delta(x) = 1/(2\pi) \int dq \exp(iqx)$, eq. (S.25) becomes

$$P_F(\mathbf{V} \rightarrow \mathbf{V}' | \mathbf{V}, t) = \frac{1}{(2\pi)^2} \int dq_1 dq_2 \exp[i(q_1 \Delta V_1 + q_2 \Delta V_2)] \int \prod_m d\eta_m e^{\Delta t \left[i q_m (f_m + \sigma_{m1}\eta_1 + \sigma_{m2}\eta_2) - \frac{\eta_m^2}{4R_m k_B T} \right]} \quad (\text{S.27})$$

$$\begin{aligned} &= \exp \left\{ -\frac{\Delta t}{4k_B T_1 T_2} \left[C_1^2 R_1 T_2 (\dot{V}_1 - f_1)^2 + C_2^2 R_2 T_1 (\dot{V}_2 - f_2)^2 \right. \right. \\ &\quad \left. \left. + 2C(\dot{V}_1 - f_1 - \dot{V}_2 + f_2)(C_1 R_1 T_2 (\dot{V}_1 - f_1) - C_2 R_2 T_1 (\dot{V}_2 - f_2)) \right. \right. \\ &\quad \left. \left. + C^2 (R_2 T_1 + R_1 T_2) (\dot{V}_1 - f_1 - \dot{V}_2 + f_2)^2 \right] \right\} \frac{X}{4\pi k_B \Delta t} \sqrt{\frac{R_1 R_2}{T_1 T_2}}; \end{aligned} \quad (\text{S.28})$$

where we have taken $\Delta V_m/\Delta t \simeq \dot{V}_m$, and exploited the fact that all the integrals in eq. (S.27) are Gaussian integrals. Similarly, for the reverse transition we obtain

$$P_R(\mathbf{V}' \rightarrow \mathbf{V}|\mathbf{V}', t + \Delta t) = \int d\eta_1 d\eta_2 \delta(\Delta V_1 + \Delta t(f_1(V'_1, V'_2) + \sigma_{11}\eta_1 + \sigma_{12}\eta_2)) \times \delta(\Delta V_2 + \Delta t(f_2(V'_1, V'_2) + \sigma_{21}\eta_1 + \sigma_{22}\eta_2)) p_1(\eta_1) p_2(\eta_2) \quad (\text{S.29})$$

$$= \exp \left\{ -\frac{\Delta t}{4k_B T_1 T_2} \left[C_1^2 R_1 T_2 (\dot{V}_1 + f_1)^2 + C_2^2 R_2 T_1 (\dot{V}_2 + f_2)^2 + 2C(\dot{V}_1 + f_1 - \dot{V}_2 - f_2)(C_1 R_1 T_2 (\dot{V}_1 + f_1) - C_2 R_2 T_1 (\dot{V}_2 + f_2)) + C^2 (R_2 T_1 + R_1 T_2) (\dot{V}_1 + f_1 - \dot{V}_2 - f_2)^2 \right] \right\} \frac{X}{4\pi k_B \Delta t} \sqrt{\frac{R_1 R_2}{T_1 T_2}}. \quad (\text{S.30})$$

We now consider the ratio between the probability of the forward and backward trajectories, and by substituting the explicit definitions of $f_1(V_1, V_2)$ and $f_2(V_1, V_2)$, as given by eqs. (S.9)-(S.10), into eqs. (S.28) and (S.30), we finally obtain

$$\log \frac{P_F(\mathbf{V} \rightarrow \mathbf{V}'|\mathbf{V}, t)}{P_R(\mathbf{V}' \rightarrow \mathbf{V}|\mathbf{V}', t + \Delta t)} = -\Delta t \left(\frac{(C_1 + C)\dot{V}_1 - C\dot{V}_2}{k_B T_1} + \frac{(C_2 + C)\dot{V}_2 - C\dot{V}_1}{k_B T_2} \right) = \Delta t \left(\frac{\dot{Q}_1}{k_B T_1} + \frac{\dot{Q}_2}{k_B T_2} \right), \quad (\text{S.31})$$

where we have exploited eq. (S.14) in order to obtain the rightmost equality. Thus, by taking a trajectory $\mathbf{V} \rightarrow \mathbf{V}'$ over an arbitrary time interval $[t, t + \tau]$, and by integrating the right hand side of eq. (S.31) over such time interval, we finally obtain

$$k_B \log \frac{P_F(\mathbf{V} \rightarrow \mathbf{V}'|\mathbf{V}, t)}{P_R(\mathbf{V}' \rightarrow \mathbf{V}|\mathbf{V}', t + \tau)} = \left(\frac{Q_1}{T_1} + \frac{Q_2}{T_2} \right) = \Delta S_{r,\tau} \quad (\text{S.32})$$

We now note that the system is in an out-of-equilibrium steady state characterized by a PDF $P_{ss}(V_1, V_2)$, and so, along any trajectory connecting two points in the phase space \mathbf{V} and \mathbf{V}' the following equality holds

$$\begin{aligned} \exp[\Delta S_{tot}/k_B] &= \exp[(\Delta S_{r,\tau} + \Delta S_{s,\tau})/k_B] \\ &= \frac{\mathcal{P}_F(\mathbf{V} \rightarrow \mathbf{V}'|\mathbf{V}, t) P_{ss}(\mathbf{V})}{\mathcal{P}_R(\mathbf{V}' \rightarrow \mathbf{V}|\mathbf{V}', t + \tau) P_{ss}(\mathbf{V}')}, \end{aligned} \quad (\text{S.33})$$

where we have exploited eq. (S.32), and the definition of $\Delta S_{s,\tau}$ in eq. (1) in the main text. Thus we finally obtain

$$\mathcal{P}_F(\mathbf{V} \rightarrow \mathbf{V}'|\mathbf{V}, t) P_{ss}(\mathbf{V}) \exp[-\Delta S_{tot}/k_B] = \mathcal{P}_R(\mathbf{V}' \rightarrow \mathbf{V}|\mathbf{V}', t + \tau) P_{ss}(\mathbf{V}') \quad (\text{S.34})$$

and summing up both sides over all the possible trajectories connecting any two points \mathbf{V}, \mathbf{V}' in the phase space, and exploiting the normalization condition of the backward probability, namely

$$\sum_{\mathbf{V}', \mathbf{V}} \mathcal{P}_R(\mathbf{V}' \rightarrow \mathbf{V}|\mathbf{V}', t + \tau) P_{ss}(\mathbf{V}') = 1, \quad (\text{S.35})$$

one obtains eq. (2). It is worth noting that the explicit knowledge of $P_{ss}(\mathbf{V})$ is not required, in order to prove eq. (2).

Finally, we note that, from a general perspective, eqs. (S.7)-(S.8) correspond to the Langevin equations of a stochastic system, whose variables V_1 and V_2 interact through non-conservative forces, and where the white noise is correlated. Therefore our proof of eq. (S.35), and thus of eq. (2) in the main text, holds in general for systems with such characteristics.

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- [1] G. Cannatà, G. Scandurra, C. Ciofin, *Rev. Scie. Instrum.* **80**, 114702 (2009).
 - [2] R. Zwanzig, *Nonequilibrium Statistical Mechanics*, Oxford University Press, Oxford, 2001.
 - [3] A. Imparato, L. Peliti, G. Pesce, G. Rusciano, A. Sasso, *Phys. Rev. E*, **76**: 050101R (2007).
 - [4] H. C. Fogedby, A. Imparato, *J. Stat. Mech.* P04005 (2012).

